Weak-type inequalities for Kantorovitch polynomials and related operators

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ABSTRACT

Continuing previous investigations concerning Bernstein polynomials, the purpose of this paper is to establish the weak-type inequality ($f\in L^p(0,1)$, $n\in\mathbb{N}$)

$$\omega_\varphi(n^{-1/2},f)\le M_p n^{-1}\sum_{k=1}^{n}\|K_kf-f\|_p$$

in terms of the Kantorovitch polynomial K_kf and the modulus of continuity ($\varphi^2(x):=x(1-x)$)

$$\omega_\varphi(t,f):=\sup_{0<h\le t}\|\Delta^2_{h\varphi}f\|_p+\sup_{0<h\le t^2}\|\Delta^2_hf\|_p.$$

Such estimates which immediately imply well-known inverse results are also obtained for the Kantorovitch version of the Szász-Mirakjan and Baskakov operators, respectively.

1. INTRODUCTION

Let $L^p(a,b)$, $1\le p<\infty$, be the space of functions f, p-th power integrable on the interval (a,b) with norm

$$\|f\|_p=\|f\|_{L^p(a,b)}:=\{\int_a^b|f(x)|^pdx\}^{1/p}.$$

For $f\in L^p(0,1)$ consider the Kantorovitch polynomial

$$K_nf:=\sum_{k=0}^{n}F_{kn}f\,p_{kn},\quad p_{kn}(x):=\binom{n}{k}x^k(1-x)^{n-k},$$

$$F_{kn}f:=(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)}f(u)du.$$

The error of this approximation process should be compared with the modulus ($\varphi^2(x):=x(1-x)$)

$$\omega_\varphi(t,f):=\sup_{0<h\le t}\|\Delta^2_{h\varphi}f\|_{L^p(h^2,1-h^2)}+\sup_{0<h\le t^2}\|\Delta^2_h f\|_{L^p(h,1-h)},$$

$$\Delta^2_h f(x):=f(x+h)-2f(x)+f(x-h).$$

Indeed, the direct estimate

$$(1.1)\qquad \|K_nf-f\|_p\le M_p[\omega_\varphi(n^{-1/2},f)+n^{-1}\|f\|_p]$$

was proved in [5]. In this paper we establish the weak-type inequality (cf. Theorem 3.2)

$$(1.2)\qquad \omega_\varphi(n^{-1/2},f)\le M_pn^{-1}\sum_{k=1}^{n}\|K_kf-f\|_p,$$

improving the inverse results of [5].

To this end, Section 2 generalizes the approach given in [7] in connection with the Bernstein polynomials. Using a characterization of $\omega_\varphi(t,f)$ in terms of a (Peetre-) K-functional (see [5]) Section 3 establishes (1.2).

This procedure may also applied to approximation processes on the positive semi-axis which is done in Section 4 for the Kantorovitch version of the Szász-Mirakjan and Baskakov operators, respectively.

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2. APPROXIMATION BY SMOOTHING FUNCTIONS

Let us commence with an extension of a lemma, given in [7].

LEMMA 2.1. *Suppose that for non-negative sequences* $\{v_n\}$, $\{\psi_n\}$ *with* $v_1=0$ *the inequality* ($s>0$, $Q\ge1$)

$$(2.1)\qquad v_n\le Q(k/n)^s v_k+M\psi_k\quad(1\le k\le n)$$

is satisfied for $n\in\mathbb{N}$. *Then one has*

$$(2.2)\qquad v_n\le M_qn^{-q}\sum_{k=1}^{n}k^{q-1}\psi_k$$

with $q=s$ *in case* $Q=1$ *and with* $0<q<s$ *else. If, additionally to* (2.1), *for some* $0<r<s$

$$(2.3)\qquad \mu_n\le(k/n)^r\mu_k+M(v_k+\psi_k)\quad(1\le k\le n)$$

holds true for a further non-negative sequence $\{\mu_n\}$ *with* $\mu_1=0$, *then*

$$(2.4)\qquad \mu_n\le M_{rs}n^{-r}\sum_{k=1}^{n}k^{r-1}\psi_k.$$

PROOF. First, let $Q>1$ and $0<q<s$. Then choose $b\ge2$ with $Qb^{-s}\le b^{-q}$. Let $n\ge b$ be given and $N\in\mathbb{N}$ be such that $b^N\le n<b^{N+1}$. Then there exist $m_k\in\mathbb{N}$,

$0 \le k \le N$, with $nb^{-k-1} < m_k \le nb^{-k}$ and

$$\psi_{m_k} \le \psi_j \quad (nb^{-k-1} < j \le nb^{-k}).$$

Setting $m_{N+1} := 1$ one therefore obtains by (2.1) that

$$v_n \le Q(m_0/n)^s v_{m_0} + M\psi_{m_0} =$$

$$= Qn^{-s} \sum_{k=0}^{N} Q^k m_k^s [v_{m_k} - Q(m_{k+1}/m_k)^s v_{m_{k+1}}] + M\psi_{m_0}$$

$$\le MQ \sum_{k=0}^{N} b^{-kq} \psi_{m_{k+1}} + M\psi_{m_0} \le MQb^q \sum_{k=0}^{N+1} b^{-kq} \psi_{m_k}$$

$$\le M_q n^{-q} \sum_{k=0}^{N+1} \sum_{nb^{-k-1} < j \le nb^{-k}} j^{q-1} \psi_j = M_q n^{-q} \sum_{j=1}^{n} j^{q-1} \psi_j .$$

The case $Q=1$, already shown in [7], follows analogously (take $b=2$). Concerning (2.4), choose $r<q<s$ (or $q=s$) so that (2.2,3) imply

$$\mu_n \le (k/n)^r \mu_k + M[M_{rs} k^{-q} \sum_{j=1}^{k} j^{q-1} \psi_j + \psi_k],$$

thus an inequality of type (2.1) but now with $Q=1$. This again delivers

$$\mu_n \le M_r n^{-r} \sum_{k=1}^{n} k^{r-1} [\psi_k + M_{rs} k^{-q} \sum_{j=1}^{k} j^{q-1} \psi_j]$$

$$= M_r n^{-r} \sum_{k=1}^{n} k^{r-1} \psi_k + M_{rs} n^{-r} \sum_{j=1}^{n} j^{q-1} \psi_j \sum_{k=j}^{n} k^{r-1-q}.$$

Since $\sum_{k=j}^{\infty} k^{r-1-q} \le M_{rs} j^{r-q}$, the lemma is proved. □

Let X be a normed linear space with norm $\|\cdot\|_X$ and let $U \subset X$ be a linear manifold with seminorm $|\cdot|_U$. Structural properties of $f \in X$ may be measured in terms of the K-functional

(2.5) $$K(t, f) = K(t, f : X, U) := \inf_{g \in U} \{ \|f-g\|_X + t|g|_U \}.$$

Let $\{T_n\}$ be a smoothing approximation process, i.e., a sequence of (bounded) linear operators on X into U.

THEOREM 2.2. *Assume that there exists a second seminorm $|\cdot|_U^*$ on U such that for $f \in X$, $g \in U$ $(0<r<s,\ n \in \mathbb{N})$*

(2.6) $$|T_n f|_U \le Mn^r \|f\|_X,$$

(2.7) $$|T_n f|_U^* \le Mn^s \|f\|_X,$$

(2.8) $$|T_n g|_U \le |g|_U + Mn^{r-s} |g|_U^*,$$

(2.9) $$|T_n g|_U^* \le M|g|_U^*.$$

Moreover, let $|T_1 f|_U = |T_1 f|_U^* = 0$ *for all* $f \in X$. *Then there holds the weak-type inequality*

$$(2.10)\qquad K(n^{-r}, f) \le M_{rs} n^{-r} \sum_{k=1}^{n} k^{r-1} \|T_k f - f\|_X \quad (f \in X).$$

PROOF. Let $f \in X$ be fixed. Setting

$$\mu_n = n^{-r} |T_n f|_U,\ \nu_n = n^{-s} |T_n f|_U^*,\ \psi_n = \|T_n f - f\|_X$$

one has by (2.6,8) that for $1 \le k \le n$

$$\begin{aligned}\mu_n &\le n^{-r} |T_n T_k f|_U + n^{-r} |T_n (T_k f - f)|_U \\ &\le n^{-r} |T_k f|_U + M n^{-s} |T_k f|_U^* + M \|T_k f - f\|_X \\ &= (k/n)^r \mu_k + M[(k/n)^s \nu_k + \psi_k] \le (k/n)^r \mu_k + M[\nu_k + \psi_k].\end{aligned}$$

Moreover, it follows in view of (2.7,9) that

$$\nu_n \le n^{-s} |T_n T_k f|_U^* + n^{-s} |T_n (T_k f - f)|_U^* \le M[(k/n)^s \nu_k + \psi_k].$$

Then Lemma 2.1 furnishes

$$|T_m f|_U \le M_{rs} \sum_{k=1}^{m} k^{r-1} \|T_k f - f\|_X \quad (m \in \mathbb{N}).$$

Now for $n \ge 2$ there exists $m \in \mathbb{N}$ such that $n/2 \le m \le n$ and

$$\|T_m f - f\|_X \le \|T_k f - f\|_X \quad (n/2 \le k \le n).$$

Therefore one obtains

$$\begin{aligned}K(n^{-r}, f) &\le \|T_m f - f\|_X + n^{-r} |T_m f|_U \\ &\le 2^r n^{-r} \sum_{n/2 \le k \le n} k^{r-1} \|T_k f - f\|_X + M_{rs} n^{-r} \sum_{k=1}^{m} k^{r-1} \|T_k f - f\|_X,\end{aligned}$$

thus the assertion. □

Note that Theorem 2.1 regains the result in [6] (for $|\cdot|_U = |\cdot|_U^*$ and $r=1$, $s=2$).

Concerning rates of convergence, let $\omega(t)$ be a positive, increasing function on $(0,\infty)$ such that for some $A > 1$

$$(2.11)\qquad \limsup_{t \to 0+} \omega(t)/\omega(t/A) < A^r.$$

Typical examples are supplied by $\omega(t) = t^\alpha |\log t|^\beta \exp\{|\log t|^\gamma\}$, $0 < \alpha < r$, $\beta \in \mathbb{R}$, $\gamma < 1$ (cf. [3]).

COROLLARY 2.3. *Under the assumptions of Theorem 2.2 suppose that* $f \in X$ *satisfies*

$$(2.12)\qquad \|T_n f - f\|_X = \mathcal{O}_f(\omega(1/n)) \quad (n \to \infty)$$

for ω subject to (2.11). *Then*

(2.13) $K(t^r, f) = \mathcal{O}_f(\omega(t)) \quad (t \to 0+)$.

PROOF. By (2.11) there exists $t_0 > 0$ and $0 < \varrho < A^r$ such that

$$\omega(t) \le \varrho\omega(t/A) \quad (0 < t \le t_0)$$

which implies for $j \in \mathbb{N}$, $0 < A^j t \le t_0$ that

$$\omega(A^j t) \le \varrho^j \omega(t).$$

Let $n \ge A/t_0$ and choose $m \in \mathbb{N}$ with $A^m/n \le t_0 < A^{m+1}/n$. Then

$$\omega(1/n) \ge \varrho^{-m}\omega(A^m/n) \ge n^{-r} t_0^{-r} \omega(t_0/A).$$

Therefore by (2.10,12)

$$K(n^{-r}, f) \le M_{rs} n^{-r}\Big(\sum_{1 \le k \le A/t_0} + \sum_{n/A^m < k \le n}\Big) k^{r-1} \| T_k f - f \|_X$$

$$= \mathcal{O}_f\Big(\omega(1/n) + n^{-r} \sum_{j=1}^{m} \sum_{n/A^j < k \le n/A^{j-1}} k^{r-1}\omega(1/k)\Big)$$

$$= \mathcal{O}_f\Big(\omega(1/n) + \sum_{j=1}^{m} \omega(A^j/n) A^{-jr}\Big)$$

$$= \mathcal{O}_f\Big(\omega(1/n)\Big[1 + \sum_{j=1}^{m} (\varrho/A^r)^j\Big]\Big) = \mathcal{O}_f(\omega(1/n)). \qquad \square$$

3. KANTOROVITCH POLYNOMIALS

Let us first establish a technical lemma, useful for L^p-approximation.

LEMMA 3.1. *Let g_{kn} be continuous on* $(0,1)$ *such that*

(3.1) $$\sum_{k=0}^{n} |g_{kn}(x)| \le \alpha_n \quad (0 < x < 1),$$

(3.2) $$\|g_{kn}\|_1 \le \beta_n \quad (0 \le k \le n).$$

Then one has for $f \in L^p(0,1)$, $1 \le p < \infty$, that

(3.3) $$\Big\| \sum_{k=0}^{n} F_{kn} f \, g_{kn} \Big\|_p \le (n+1)^{1/p} \alpha_n^{1-1/p} \beta_n^{1/p} \|f\|_p.$$

PROOF. Set $\chi_n(x,u) := (n+1) g_{kn}(x)$ in case $k/(n+1) < u < (k+1)/(n+1)$ and $0 \le k \le n$. In view of (3.1,2) it follows that

$$\int_0^1 |\chi_n(x,u)| du \le \alpha_n, \quad \int_0^1 |\chi_n(x,u)| dx \le (n+1)\beta_n.$$

Then by Hölder's inequality

$$\left|\sum_{k=0}^{n} F_{kn}f\, g_{kn}(x)\right| = \left|\int_0^1 f(u)\chi_n(x,u)du\right|$$

$$\leq \alpha_n^{1-1/p}\left(\int_0^1 |f(u)|^p |\chi_n(x,u)| du\right)^{1/p}.$$

Moreover, Fubini's theorem implies (3.3) since

$$\left\|\sum_{k=0}^{n} F_{kn}f\, g_{kn}\right\|_p^p \leq \alpha_n^{p-1}\int_0^1 |f(u)|^p \int_0^1 |\chi_n(x,u)| dx du$$

$$\leq (n+1)\alpha_n^{p-1}\beta_n \|f\|_p^p. \qquad \square$$

The proof of (1.2) essentially depends on the estimate (cf. [5])

$$(3.4) \qquad \omega_\varphi(t,f) \leq M_p K(t^2,f) \quad (t>0, f\in L^p(0,1))$$

where K is defined by (2.5) with $X = L^p(0,1)$ and

$$U := \{g \in L^p(0,1) : \varphi^2 g'' \in L^p(0,1)\}, \; |g|_U := \|\varphi^2 g''\|_p.$$

THEOREM 3.2. *For $f \in L^p(0,1)$, $1 \leq p < \infty$, there holds true the weak-type inequality*

$$(3.5) \qquad \omega_\varphi(n^{-1/2}, f) \leq M_p n^{-1} \sum_{k=1}^{n} \|K_k f - f\|_p.$$

PROOF. To apply Theorem 2.2 set $T_n = K_n$ and $|g|_U^* = \|g''\|_p$. Now (cf. [2] for Bernstein polynomials)

$$\varphi^2 K_n'' f = \sum_{k=0}^{n} F_{kn} f\, g_{kn},$$

$$g_{kn}(x) := n[(n-1)(k/n - x)^2 - \varphi^2(k/n)] p_{kn}(x)/\varphi^2(x).$$

Then (3.1) with $\alpha_n = 2n$ is obvious. Moreover, in view of $\|p_{kn}\|_1 = (n+1)^{-1}$ one obtains

$$\left\|\varphi^2\left(\frac{k}{n}\right) p_{kn}/\varphi^2\right\|_1 = \left(1 - \frac{1}{n}\right)\|p_{k-1,n-2}\|_1 = \frac{1}{n},$$

$$\left\|\left(\frac{k}{n} - x\right)^2 p_{kn}(x)/\varphi^2(x)\right\|_1 = \int_0^1 \left[-\left(1 - \frac{1}{n}\right) p_{k-1,n-2}(x) - p_{kn}(x)\right.$$

$$\left. + p_{k-1,n-1}(x) + p_{k,n-1}(x)\right] dx$$

$$= -\frac{1}{n} - \frac{1}{n+1} + \frac{2}{n} = \frac{1}{n(n+1)}$$

so that $\|g_{kn}\|_1 \le 2 = \beta_n$. Then (2.6) with $r=1$ follows since by (3.3)

$$\|\varphi^2 K_n'' f\|_p \le (n+1)^{1/p}(2n)^{1-1/p}2^{1/p}\|f\|_p \le 4n\|f\|_p.$$

To establish (2.7) with $s=2$ apply the representation

$$\text{(3.6)} \qquad \begin{cases} K_n'' f = n(n-1)\sum_{k=0}^{n-2} \Delta^2 F_{k+1,n} f\, p_{k,n-2} = n^2 \varphi^{-2} \sum_{k=1}^{n-1} \varphi^2\left(\frac{k}{n}\right)\Delta^2 F_{kn} f\, p_{kn}, \\ \Delta^2 F_{kn} f = F_{k-1,n} f - 2F_{kn} f + F_{k+1,n} f, \end{cases}$$

which delivers by Lemma 3.1

$$\|K_n'' f\|_p \le 4n(n+1)^{1/p}(n-1)^{1-1/p}\|f\|_p \le 12n^2\|f\|_p.$$

Concerning (2.8) let $g \in U$. Then $(1 \le k \le n-1)$

$$\text{(3.7)} \qquad \Delta^2 F_{kn} g = (n+1) \iiint_{-1/2(n+1)}^{1/2(n+1)} g''\left(\frac{k+1/2}{n+1} + u + s + t\right) du\, ds\, dt.$$

With $y = \dfrac{k+1/2}{n+1} + u + s + t$ one has

$$\left|\frac{k}{n} - y\right| \le \frac{|k - n/2|}{n(n+1)} + \frac{3/2}{n+1} \le \frac{2}{n+1},$$

$$\varphi^2\left(\frac{k}{n}\right) = \varphi^2(y) + \left(\frac{k}{n} - y\right)(1-2y) - \left(\frac{k}{n} - y\right)^2 \le \varphi^2(y) + \frac{2}{n+1}$$

so that with $G_n(v) := \left[\varphi^2(v) + \dfrac{2}{n+1}\right]|g''(v)|$

$$\varphi^2\left(\frac{k}{n}\right)|\Delta^2 F_{kn} g| \le (n+1) \iiint_{-1/2(n+1)}^{1/2(n+1)} G_n\left(\frac{k+1/2}{n+1} + u + s + t\right) du\, ds\, dt$$

$$= F_{kn} H_n,$$

$$H_n(v) := \begin{cases} \iint_{-1/2(n+1)}^{1/2(n+1)} G_n(v+s+t)\, ds\, dt, & \dfrac{1}{n+1} \le v \le \dfrac{n}{n+1} \\ 0 & \text{, else.} \end{cases}$$

Then Hölder's inequality yields

$$H_n(v) \le (n+1)^{2/p-2}\left\{\iint_{-1/2(n+1)}^{1/2(n+1)} G_n^p(v+s+t)\, ds\, dt\right\}^{1/p},$$

$$\|H_n\|_p \le (n+1)^{2/p-2}\left\{\iint_{-1/2(n+1)}^{1/2(n+1)} \left[\int_{1/(n+1)}^{n/(n+1)} G_n^p(v+s+t)\, dv\right] ds\, dt\right\}^{1/p}$$

$$\le (n+1)^{-2}\|G_n\|_p \le (n+1)^{-2}\left[\|\varphi^2 g''\|_p + \frac{2}{n+1}\|g''\|_p\right],$$

thus (2.8) since by (3.3,6)

$$\|\varphi^2 K_n'' g\|_p \leq n^2 \|K_n H_n\|_p \leq \|\varphi^2 g''\|_p + \frac{2}{n}\|g''\|_p.$$

Moreover, (2.9) follows analogously (take $G_n = g''$ to estimate $\Delta^2 F_{kn} g$) so that Theorem 2.2 in connection with $K_1'' f = 0$ and (3.4) establishes the assertion. □

Concerning rates of convergence Corollary 2.3 in connection with (1.1) delivers

COROLLARY 3.3. *Let ω satisfy* (2.11) *with* $r=1$. *Then for* $f \in L^p(0,1)$ *the condition*

$$\|K_n f - f\|_p = \mathcal{O}_f(\omega(1/n)) \quad (n \to \infty)$$

is necessary and sufficient for

$$\omega_\varphi(t,f) = \mathcal{O}_f(\omega(t^2)) \quad (t \to 0+).$$

4. APPROXIMATION ON THE POSITIVE SEMI-AXIS

For $f \in L^p(0,\infty)$ consider the Kantorovitch version

$$S_n f := \sum_{k=0}^{\infty} F_{k,n-1} f\, s_{kn}, \quad s_{kn}(x) := e^{-nx}\frac{(nx)^k}{k!},$$

$$V_n f := \sum_{k=0}^{\infty} F_{k,n-1} f\, b_{kn}, \quad b_{kn}(x) := \binom{n+k-1}{k} x^k (1+x)^{-n-k}$$

of the Szász-Mirakjan and Baskakov operator, respectively. The corresponding modulus can be expressed by (cf. [5])

$$\omega_\varphi(t,f) := \sup_{0<h\leq t} \|\Delta_{h\varphi}^2 f\|_{L^p(ah^2,\infty)} + \sup_{0<h\leq t^2} \|\Delta_h^2 f\|_{L^p(h,1/2)},$$

$$(4.1) \qquad \begin{cases} \varphi^2(x) = x, \ a=1 \text{ if } T_n = S_n \\ \varphi^2(x) = x(1+x), \ a=2 \text{ if } T_n = V_n. \end{cases}$$

Again, this modulus is equivalent to the K-functional defined by (2.5) with

$$U := \{g \in L^p(0,\infty) : \varphi^2 g'' \in L^p(0,\infty)\}, \ |g|_U := \|\varphi^2 g''\|_p,$$

so that one similarly obtains

THEOREM 4.1. *With the notation* (4.1) *one has for* $f \in L^p(0,\infty)$, $1 \leq p < \infty$,

$$\omega_\varphi(n^{-1/2}, f) \leq M_p \begin{cases} n^{-1}[\sum_{k=1}^{n} \|S_k f - f\|_p + \|f\|_p] \\ n^{-1}[\sum_{k=1}^{n} \|V_k f - f\|_p + \|f\|_p]. \end{cases}$$

PROOF. Let us follow the lines of the proof of Theorem 3.2. First of all, note that Lemma 3.1 also holds true for the interval $(0,\infty)$ and for infinite series. Now set $T_1=0$ (then (2.6–9) are trivial) and for $n\geq 2$ alternatively

$$\begin{cases} T_n=S_n, \ |g|_U^*=\|g''\|_p, & q_{kn}=s_{kn} \\ T_n=V_n, \ |g|_U^*=\|g''\|_p+\|\varphi^2 g''\|_p, & q_{kn}=b_{kn}. \end{cases}$$

Then one has (cf. [1])

$$\varphi^2 T_n'' f=\sum_{k=0}^{\infty} F_{k,n-1} f\, q_{kn}=n^2\sum_{k=1}^{\infty}\varphi^2\left(\frac{k}{n}\right)\Delta^2 F_{k,n-1} f\, q_{kn},$$

$$g_{kn}(x)=n\varphi^{-2}(x)q_{kn}(x)\cdot\begin{cases} \left[n\left(\frac{k}{n}-x\right)^2-\varphi^2\left(\frac{k}{n}\right)\right], & T_n=S_n \\ \left[(n+1)\left(\frac{k}{n}-x\right)^2-\varphi^2\left(\frac{k}{n}\right)\right], & T_n=V_n \end{cases}$$

which implies (2.6,7) (cf. [4]). Also (2.8) follows since for $|k/n-y|\leq 2/n$

$$\varphi^2\left(\frac{k}{n}\right)\leq\begin{cases} \varphi^2(y)+\frac{2}{n}, & T_n=S_n \\ \varphi^2(y)+\frac{2}{n}(1+2y)+\frac{4}{n^2}\leq\varphi^2(y)+\frac{4}{n}(1+\varphi^2(y)), & T_n=V_n. \end{cases}$$

Analogously, one establishes (2.9) in view of

$$\varphi^2\left(\frac{k}{n}\right)q_{kn}(x)=\begin{cases} \varphi^2(x)q_{k-1,n}(x), & q_{kn}=s_{kn} \\ \frac{n+1}{n}\varphi^2(x)q_{k-1,n+2}(x), & q_{kn}=b_{kn}. \end{cases}$$

Hence Theorem 2.2 and the equivalence of the K-functional with ω_φ yield the assertion. □

Let us finally apply these weak-type inequalities in connection with the direct estimates (cf. [5])

$$\left.\begin{array}{l} \|S_n f-f\|_p \\ \|V_n f-f\|_p \end{array}\right\}\leq M_p[\omega_\varphi(n^{-1/2},f)+n^{-1}\|f\|_p].$$

COROLLARY 4.2. *Let ω satisfy* (2.11) *with $r=1$. Then for $f\in L^p(0,\infty)$ the condition*

$$\left.\begin{array}{l} \|S_n f-f\|_p \\ \|V_n f-f\|_p \end{array}\right\}=\mathcal{O}_f(\omega(1/n)) \quad (n\to\infty),$$

respectively, is necessary and sufficient for (*cf.* (4.1))

$$\omega_\varphi(t,f)=\mathcal{O}_f(\omega(t^2)) \quad (t\to 0+).$$

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